STUDIA UNIVERSITATIS BABEŞ-BOLYAI OECONOMICA
VOLUME 65, ISSUE 1, 2020, pp. 20-38
DOI: 10.2478/subboec-2020-0002

A KEYNESIAN APPROACH TO FISCAL POLICY FOR FULL EMPLOYMENT AND CONTINUOUS TIME DEBT DYNAMICS

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Abstract: It is widely argued that public debt is a burden on the future generations. We analyze another aspect of public debt as an economic stimulus program, that is, the measure to realize full employment from an under-employment state. Using a continuous time version of a dynamic analysis of debt-to-GDP ratio we show that a fiscal policy to realize full employment from a state of under-employment can reduce the debt-to-GDP ratio. More precisely we show that the larger the extra growth rate (increasing rate) of real GDP by a fiscal policy is, the smaller the debt-to-GDP ratio at the time when full employment is realized is. Also we show that even if the marginal propensity to consume is very small (including zero), an aggressive fiscal policy can realize full employment without increasing the debt-to-GDP ratio. Further, we consider a condition to realize full employment from a state of under-employment within one year without increasing debt-to-GDP ratio.

JEL classification: E12, E24, E62

Keywords: full employment, debt-to-GDP ratio, continuous time debt dynamics

1. Introduction

It is widely argued that public debt is a burden on the future generations. There are many studies about the burden of public debt in a full employment state and an under-employment state such as Lerner (1944), Diamond (1965), Barro (1974), Rankin (1986), Sen (2002), Ono (2011), Tanaka (2013) and Otaki (2015). Otaki (2015), by using an overlapping generations model, showed that public debt lowers the future generation's welfare in the situation of under-employment. Using a simple textbook multiplier model, Ono (2011) showed that an increase in a wasteful public spending under a balanced budget or a loan budget without money illusion raises GDP, but it

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is not effective because the household's consumption does not change. J. Tanaka (2013) studied the welfare effects of fiscal policies in an under-employment economy using a fixed price non-Walrasian overlapping generations model[1], and showed that in all three cases, (1) wasteful spending, (2) an inter-generational income transfer, and (3) an intra-generational income transfer between different groups of households, there is no burden of public debt. These are studies of the welfare effects of public debt in a situation of under-employment. On the other hand, for example, Sen (2002) analysed the welfare effects of public debt in a situation of full employment.

The focus of this paper is different from that of these studies. We analyze another aspect of public debt as an economic stimulus program, that is, the measure to realize full employment from an under-employment state. Watts and Sharpe (2016) presented a discrete time version of dynamic analysis of debt-to-GDP ratio, and showed that an aggressive fiscal policy can reduce the debt-to-GDP ratio. Generalizing their model we present a continuous time version of a dynamic analysis of debt-to-GDP ratio, and examine the effects of a fiscal policy which realizes full employment from a state of under-employment or with deflationary GDP gap[2]. Under-employment state arises due to aggregate demand shortage. As discussed in Mitchell, Wray, and Watts (2019) sustained unemployment imposes significant social costs such as loss of current national output and income, skill loss, and so on. Therefore, it is valuable that full employment is realized in the short term.

We consider time required to realize full employment from a state of under-employment, and examine the debt-to-GDP ratio at the time when full employment is realized. The government increases its expenditure to accelerate the economic growth until full employment is realized. The extra growth rate (increasing rate) of the government expenditure over the ordinary growth rate (the growth rate of the full employment real GDP) depends on the target growth rate of real GDP over ordinary growth rate, the share of the government expenditure in real GDP, and the magnitude of multiplier effects. We show that a fiscal policy to realize full employment from a state of under-employment can reduce the debt-to-GDP ratio, and that the larger the extra growth rate (increasing rate) of real GDP by a fiscal policy is, the smaller the debt-to-GDP ratio at the time when full employment is realized is. Also we show that even if the marginal propensity to consume is very small (including zero), an aggressive fiscal policy can realize full employment without increasing the debt-to-GDP ratio.

In the next section we consider a steady state of continuous time debt dynamics, and analyze the effects of a fiscal policy to realize full employment. In Section 3 we present some graphical simulations based on plausible assumptions of variables. In Appendix we present a derivation of multiplier in a dynamic overlapping generations model according to Otaki (2007, 2009).

Let g be the growth rate of the full employment real GDP, ρ be the extra growth rate of real GDP over g by a fiscal policy (the growth rate of real GDP is

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[1] According to J. Tanaka (2013) the first attempt to present a non-Walrasian overlapping generations model is Rankin (1986). He examined the effects of a permanent increase in public debt stock on capital accumulation.

[2] In another paper we have presented an analysis and a simulation of fiscal policy for full employment using a discrete time version of debt dynamics.

$g+\rho$) in a state of under-employment, and γ be the extra growth rate of the government expenditure over g by a fiscal policy (the growth rate of the government expenditure is $g+\gamma$). The main results are as follows.

1. The larger the value of ρ is, the faster the full employment state is realized. (Figure 1)

2. The larger the value of ρ is, the smaller the debt-to-GDP ratio at the time when full employment is realized is, that is, the more aggressive the fiscal policy is, the smaller the debt-to-GDP ratio at the time when full employment is realized is. (Figure 3)

The reason for this result is as follows. The smaller the value of ρ is, the longer the time we need to realize full employment is. On the other hand, as stated in 5 below (Proposition 1), the share of the government expenditure in real GDP at the time when full employment is realized does not depend on ρ. Therefore, when ρ is small, the accumulated budget deficit including burden of interest is large.

3. When the value of ρ is larger than the critical value, the fiscal policy to realize full employment reduces the debt-to-GDP ratio. (Figure 4)

4. By a fiscal policy, first the debt-to-GDP ratio increases, and then it decreases. (Figure 5 and 6)

5. The share of the government expenditure in real GDP at the time when full employment is realized does not depend on the values of ρ and γ. (Proposition 1)

6. Even if the marginal propensity to consume is very small, an aggressive fiscal policy can realize full employment without increasing debt-to-GDP ratio (Subsection 3.10).

The main conclusion of this paper is that full employment can be realized by an aggressive fiscal policy with smaller debt-to-GDP ratio than before the fiscal policy.

An increase in the government expenditure may induce a rise in the interest rate. Since the higher the interest rate is, the larger the debt-to-GDP ratio is (Subsection 3.9), we need an appropriate monetary policy which maintains the low interest rate[3].

2. Continuous time debt dynamics

We consider a continuous time version of debt dynamics. The variables are as follows.

c: marginal propensity to consume, $0<c<1$[4],
τ: tax rate, $0<\tau<1$,

$$\beta=1-c(1-\tau),\ 0<\beta<1,$$

[3] Of course, a rise in the interest rate may reduce the investment.
[4] About consumption functions in a dynamic Keynesian model please see Otaki (2007, 2009). In Appendix we present a derivation of multiplier by an overlapping generations model of consumption.

$Y(0)$: real GDP at time 0,
$Y(t)$: real GDP at time t, $t \geq 0$,
$Y_m(0)$: full employment real GDP at time 0,
$Y_m(t)$: full employment real GDP at time t, $t \geq 0$,

$$\zeta = \frac{Y_m(0)}{Y(0)}, \ \zeta > 1,$$

$\tilde{t}$: the time at which full employment is realized, $\tilde{t} > 0$,
$G(0)$: government expenditure at time 0,
$G(t)$: government expenditure at time t,
$T(0)$: tax revenue at time 0,
$T(t)$: tax revenue at time t,

$$\alpha = \frac{G(0)}{Y(0)},$$

$B(0)$: government budget surplus at time 0,
$B(t)$: government budget surplus at time t,

$$b(0) = \frac{B(0)}{Y(0)},$$
$$b(t) = \frac{B(t)}{Y(t)},$$

$D(0)$: government debt at time 0,
$D(t)$: government debt at time t,

$$d(0) = \frac{D(0)}{Y(0)},$$
$$d(t) = \frac{D(t)}{Y(t)},$$

d^*: the steady state value of $d(t)$,
g: the growth rate of the full employment real GDP, $g > 0$,
ρ: the extra growth rate of real GDP by a fiscal policy, $\rho > 0$,
γ: the extra growth rate of the government expenditure by a fiscal policy, $\gamma > 0$,
r: interest rate.

The unit of time is a year. We assume $g + \rho > r$.

Using approximations of exponential functions, we also show the following result (Proposition 2).

If the full employment state is realized within one year, $d(0)$ and $b(0)$ have the steady state values, and the propensity to consume c satisfies the following condition

$$c > 1 - \frac{2}{1-\tau} d(0),$$

then the debt-to-GDP ratio at the time when the full employment state is realized is smaller than that before the fiscal policy. If $d(0) > 0.5$, this condition is always satisfied for even very small (including zero) propensity to consume.

2.1. A steady state

First we examine a steady state of debt dynamics. At the steady state

$$Y(t) = e^{gt}Y(0), \qquad G(t) = e^{gt}G(0), \qquad T(t) = e^{gt}T(0).$$

Thus,

$$B(t) = T(t) - G(t) = e^{gt}B(0).$$

The derivative of $D(t)$ with respect to t is

$$D'(t) = rD(t) - B(t).$$

$D(t)$ is calculated as

$$D(t) = e^{rt}D(0) - \int_0^t e^{r(t-s)}B(s)ds = e^{rt}D(0) - \int_0^t e^{r(t-s)}e^{gs}B(0)ds$$
$$= e^{rt}D(0) - e^{rt}B(0)\int_0^t e^{(g-r)s}ds = e^{rt}D(0) - e^{rt}B(0)\left[\frac{e^{(g-r)s}}{g-r}\right]_0^t$$
$$= e^{rt}D(0) - e^{rt}B(0)\frac{e^{(g-r)t}-1}{g-r}.$$

$e^{r(t-s)}$ denotes a burden of interest between s and t. Since $Y(t) = e^{gt}Y(0)$,

$$\frac{D(t)}{Y(t)} = e^{(r-g)t}\frac{D(0)}{Y(0)} - e^{(r-g)t}\frac{B(0)}{Y(0)}\frac{e^{(g-r)t}-1}{g-r}.$$

Therefore, the debt-to-GDP ratio at time t is obtained as follows.

$$d(t) = e^{(r-g)t}d(0) - e^{(r-g)t}b(0)\frac{e^{(g-r)t}-1}{g-r}.$$

At the steady state

$$d(t) = d(0) = d^*.$$

Then,

$$d^* = \frac{0}{1-e^{(r-g)t}}\left[b(0)\frac{1-e^{(r-g)t}}{r-g}\right] = \frac{b(0)}{r-g}. \qquad (1)$$

2.2. Fiscal policy for full employment

We assume that there exists a deflationary GDP gap, that is, $Y(0)$ is smaller than the full employment real GDP, $Y_m(0)$, at time 0. Then, $\zeta > 1$. Since $Y_m(t)$ increases at the rate g,

$$Y_m(t) = e^{gt}Y_m(0).$$

The government increases the growth rate of its expenditure from g to $g+\gamma$ to increase the growth rate of real GDP from g to $g+\rho$ so as to realize full employment. Then,

$$Y(t)=e^{(g+\rho)t}Y(0).$$

Suppose that at time $\tilde{t}$

$$e^{(g+\rho)\tilde{t}}Y(0)=e^{g\tilde{t}}Y_m(0),$$

that is, full employment is realized at $\tilde{t}$. Then, we have

$$e^{\rho\tilde{t}}=\zeta.$$

$\tilde{t}$ is obtained as follows.

$$\tilde{t}=\frac{\ln\zeta}{\rho}. \tag{2}$$

The larger the value of ρ is, the faster the full employment state is realized. Since $G(t)$ increases at the rate $g+\gamma$,

$$G(t)=e^{(g+\gamma)t}G(0).$$

We examine the relation between ρ and γ. The increase in real GDP over the ordinary growth is brought by the *multiplier effect* of an increase in the government expenditure over the ordinary growth. Therefore, we have the following relation

$$\frac{1}{\beta}\left[e^{(g+\gamma)\tilde{t}}-e^{g\tilde{t}}\right]G(0)=\left[e^{(g+\rho)\tilde{t}}-e^{g\tilde{t}}\right]Y(0).$$

This means

$$\frac{1}{\beta}\left(e^{\gamma\tilde{t}}-1\right)G(0)=\left(e^{\rho\tilde{t}}-1\right)Y(0).$$

And so

$$\frac{\alpha}{\beta}\left(e^{\gamma\tilde{t}}-1\right)=e^{\rho\tilde{t}}-1,$$

or

$$e^{\gamma\tilde{t}}=\frac{\beta}{\alpha}\left(e^{\rho\tilde{t}}-1\right)+1.$$

Since $\zeta=e^{\rho\tilde{t}}$,

$$e^{\gamma\tilde{t}}=\frac{\beta}{\alpha}(\zeta-1)+1.$$

Thus,

$$\gamma=\frac{\rho\ln\left[\frac{\beta}{\alpha}(\zeta-1)+1\right]}{\ln\zeta}. \tag{3}$$

$B(t)$ is the sum of the budget surplus growing by g from $B(0)$ and the budget surplus brought by the fiscal policy. It is written as

$$B(t) = e^{gt}B(0) + \tau\left(e^{(g+\rho)t} - e^{gt}\right)Y(0) - \left(e^{(g+\gamma)t} - e^{gt}\right)G(0).$$

The derivative of $D(t)$ with respect to t is

$$D'(t) = rD(t) - B(t)$$
$$= rD(t) - e^{gt}B(0) - \tau\left(e^{(g+\rho)t} - e^{gt}\right)Y(0) + \left(e^{(g+\gamma)t} - e^{gt}\right)\alpha Y(0).$$

Therefore,

$$D(t) = e^{rt}D(0) - B(0)\int_0^t e^{(t-s)r}e^{gs}ds - \tau Y(0)\int_0^t e^{(t-s)r}\left(e^{(g+\rho)s} - e^{gs}\right)ds$$
$$+\alpha Y(0)\int_0^t e^{(t-s)r}\left(e^{(g+\gamma)s} - e^{gs}\right)ds$$
$$= e^{rt}D(0) - e^{rt}B(0)\int_0^t e^{(g-r)s}ds - e^{rt}\tau Y(0)\int_0^t \left(e^{(g+\rho-r)s} - e^{(g-r)s}\right)ds$$
$$+e^{rt}\alpha Y(0)\int_0^t \left(e^{(g+\gamma-r)s} - e^{(g-r)s}\right)ds.$$

Since

$$Y(t) = e^{(g+\rho)t}Y(0),$$

we get

$$d(t) = e^{(r-g-\rho)t}d(0) - e^{(r-g-\rho)t}b(0)\int_0^t e^{(g-r)s}ds$$
$$-e^{(r-g-\rho)t}\tau\int_0^t \left(e^{(g+\rho-r)s} - e^{(g-r)s}\right)ds + e^{(r-g-\rho)t}\alpha\int_0^t \left(e^{(g+\gamma-r)s} - e^{(g-r)s}\right)ds$$
$$= e^{(r-g-\rho)t}d(0) - e^{(r-g-\rho)t}b(0)\left[\frac{e^{(g-r)s}}{g-r}\right]_0^t - e^{(r-g-\rho)t}\tau\left[\frac{e^{(g+\rho-r)s}}{g+\rho-r} - \frac{e^{(g-r)s}}{g-r}\right]_0^t$$
$$+e^{(r-g-\rho)t}\alpha\left[\frac{e^{(g+\gamma-r)s}}{g+\gamma-r} - \frac{e^{(g-r)s}}{g-r}\right]_0^t$$
$$= e^{(r-g-\rho)t}d(0) - e^{(r-g-\rho)t}b(0)\left[\frac{e^{(g-r)t}-1}{g-r}\right]$$
$$- e^{(r-g-\rho)t}\tau\left[\frac{e^{(g+\rho-r)t}-1}{g+\rho-r} - \frac{e^{(g-r)t}-1}{g-r}\right]$$
$$+e^{(r-g-\rho)t}\alpha\left[\frac{e^{(g+\gamma-r)t}-1}{g+\gamma-r} - \frac{e^{(g-r)t}-1}{g-r}\right].$$

Thus,

$$d(t) = e^{(r-g-\rho)t}d(0) - b(0)\left[\frac{e^{-\rho t} - e^{(r-g-\rho)t}}{g-r}\right] \quad (4)$$
$$-\tau\left[\frac{1-e^{(r-g-\rho)t}}{g+\rho-r} - \frac{e^{-\rho t} - e^{(r-g-\rho)t}}{g-r}\right] + \alpha\left[\frac{e^{(\gamma-\rho)t} - e^{(r-g-\rho)t}}{g+\gamma-r} - \frac{e^{-\rho t} - e^{(r-g-\rho)t}}{g-r}\right].$$

Let $t = \tilde{t}$. Then,

$$d(\tilde{t}) = e^{-\rho\tilde{t}} e^{(r-g)\tilde{t}} d(0) - e^{-\rho\tilde{t}} b(0) \left[\frac{1-e^{(r-g)\tilde{t}}}{g-r}\right] \tag{5}$$
$$-e^{-\rho\tilde{t}}\tau\left[\frac{e^{\rho\tilde{t}} - e^{(r-g)\tilde{t}}}{g+\rho-r} - \frac{1-e^{(r-g)\tilde{t}}}{g-r}\right] + e^{-\rho\tilde{t}}\alpha\left[\frac{e^{\gamma\tilde{t}} - e^{(r-g)\tilde{t}}}{g+\gamma-r} - \frac{1-e^{(r-g)\tilde{t}}}{g-r}\right]$$
$$= \frac{1}{\zeta}\left\{e^{(r-g)\tilde{t}} d(0) - b(0)\left[\frac{1-e^{(r-g)\tilde{t}}}{g-r}\right] - \tau\left[\frac{\zeta - e^{(r-g)\tilde{t}}}{g+\rho-r} - \frac{1-e^{(r-g)\tilde{t}}}{g-r}\right]\right.$$
$$\left. +\alpha\left[\frac{e^{\gamma\tilde{t}} - e^{(r-g)\tilde{t}}}{g+\gamma-r} - \frac{1-e^{(r-g)\tilde{t}}}{g-r}\right]\right\}.$$

From (5),

$$d(\tilde{t}) - d(0) = \frac{1}{\zeta}\left\{\left[e^{(r-g)\tilde{t}} - \zeta\right] d(0) - b(0)\left[\frac{1-e^{(r-g)\tilde{t}}}{g-r}\right]\right. \tag{6}$$
$$\left. -\tau\left[\frac{\zeta - e^{(r-g)\tilde{t}}}{g+\rho-r} - \frac{1-e^{(r-g)\tilde{t}}}{g-r}\right] + \alpha\left[\frac{e^{\gamma\tilde{t}} - e^{(r-g)\tilde{t}}}{g+\gamma-r} - \frac{1-e^{(r-g)\tilde{t}}}{g-r}\right]\right\}.$$

Because $e^{(r-g)\tilde{t}} - \zeta = e^{(r-g)\tilde{t}} - e^{\rho\tilde{t}} < 0$ by $g + \rho > r$ or $r - g < \rho$, (6) is decreasing with respect to $d(0)$. γ is obtained from (3), and $\tilde{t}$ is obtained from (2).

$\alpha = \frac{G(0)}{Y(0)}$ is the share of the government expenditure in real GDP at time 0. Real GDP grows at the rate $g + \rho$, on the other hand the government expenditure grows at the rate $g + \gamma$, and $\gamma > \rho$. The larger the values of ρ and γ are, the smaller the time necessary for realization of full employment is. The value of α at $\tilde{t}$ is denoted by

$$\alpha(\tilde{t}) = \frac{G(\tilde{t})}{Y(\tilde{t})} = \frac{e^{(g+\gamma)\tilde{t}}}{e^{(g+\rho)\tilde{t}}}\alpha = e^{(\gamma-\rho)\tilde{t}}\alpha.$$

From (2) and (3), we get

$$\alpha(\tilde{t}) = e^{\left(\frac{\ln\left[\frac{\beta}{\alpha}(\zeta-1)+1\right]}{\ln\zeta} - 1\right)\rho\frac{\ln\zeta}{\rho}}\alpha = \frac{\beta(\zeta-1)+\alpha}{\zeta}.$$

This is constant, that is, it does not depend on ρ and γ. We have shown the following result.

Proposition 1 *The share of the government expenditure in real GDP at the time when full employment is realized does not depend on the values of ρ and γ.*

If $d(0)$ and $b(0)$ have the steady state values, that is, $b(0) = (r-g)d(0)$, then (6) is rewritten as

$$d(\tilde{t}) - d(0)|_{b(0)=(r-g)d(0)} = \frac{1}{\zeta}\{(1-\zeta)d(0)$$
$$\left. -\tau\left[\frac{\zeta - e^{(r-g)\tilde{t}}}{g+\rho-r} - \frac{1-e^{(r-g)\tilde{t}}}{g-r}\right] + \alpha\left[\frac{e^{\gamma\tilde{t}} - e^{(r-g)\tilde{t}}}{g+\gamma-r} - \frac{1-e^{(r-g)\tilde{t}}}{g-r}\right]\right\}.$$

Suppose $\tilde{t}=1$, that is, full employment is realized within one year. Then, with $e^{\rho}=\zeta$,

$$d(\tilde{t})-d(0)|_{b(0)=(r-g)d(0)}=\frac{1}{\zeta}\{(1-\zeta)d(0)$$
$$-\tau\left[\frac{e^{\rho}-e^{r-g}}{g+\rho-r}-\frac{1-e^{(r-g)}}{g-r}\right]+\alpha\left[\frac{e^{\gamma}-e^{(r-g)}}{g+\gamma-r}-\frac{1-e^{(r-g)}}{g-r}\right]\Bigg\}.$$

We use the following approximation for exponential functions for small x;

$$e^{x}=1+x+\frac{1}{2}x^{2}.$$

Then, we obtain

$$d(\tilde{t})-d(0)|_{b(0)=(r-g)d(0)}=\frac{1}{\zeta}\{(1-\zeta)d(0)$$
$$-\frac{1}{2}\tau[(\rho+r-g)-(r-g)]+\frac{1}{2}\alpha[(\gamma+r-g)-(r-g)]\Big\}$$
$$=\frac{1}{\zeta}\left[(1-\zeta)d(0)-\frac{1}{2}(\tau\rho-\alpha\gamma)\right]$$

We apply the following approximation to (3),

$$\ln x=x-1,$$

and apply the following approximation to $\zeta=e^{\rho}$,

$$e^{\rho}=1+\rho.$$

Then, we get

$$\alpha\gamma=\beta\rho=[1-c(1-\tau)]\rho,\qquad 1-\zeta=-\rho.$$

Thus,

$$d(\tilde{t})-d(0)|_{b(0)=(r-g)d(0)}=\frac{\rho}{\zeta}\left[-d(0)+\frac{1}{2}(1-\tau)(1-c)\right]$$

If

$$\frac{1}{2}(1-\tau)(1-c)<d(0),$$

we have $d(\tilde{t})-d(0)<0$. This condition is rewritten as

$$c>1-\frac{2}{1-\tau}d(0).$$

If $d(0)>0.5$, this is always satisfied even if $c=0$. We have shown the following result.

Proposition 2 *If the full employment state is realized within one year,* $d(0)$ *and* $b(0)$ *have the steady state values, and the propensity to consume* c *satisfies the following condition*

$$c>1-\frac{2}{1-\tau}d(0),$$

the debt-to-GDP ratio at the time when the full employment state is realized is smaller than that before the fiscal policy. If $d(0) > 0.5$, this condition is always satisfied for even very small (including zero) propensity to consume.

3. Graphical simulations

We present some simulation results. Assume the following values for the variables.

$$c = 0.5,\ \tau = 0.25,\ \alpha = 0.3,\ g = 0.025,\ r = 0.015,\ b(0) = -0.015 \text{ and } \zeta = 1.15.$$

We assume that g and r are constant, and $g > r$[5]. However, in Subsection 3.9 we examine a case where $r > g$. We do not assume that $d(0)$ and $b(0)$ have the steady state values described in (1). But, in Subsection 3.11 we consider a case where $d(0)$ and $b(0)$ have the steady state values.

3.1. Relation between ρ and $\tilde{t}$

In addition to the above assumptions we assume $d(0) = 0.45$, which is the debt-to-GDP ratio at the tome 0. Figure 1 represents the relation between ρ and $\tilde{t}$ which is the time at which full employment is realized. ρ is the extra growth rate of real GDP over g by a fiscal policy. As (2) suggests, the larger the value of ρ is, the smaller the value of $\tilde{t}$ is, that is, the faster the full employment state is realized.

Therefore, the more aggressive the fiscal policy is, the faster full employment is realized. For example, when $\rho = 0.05$, $\tilde{t} \approx 2.7$, when $\rho = 0.1$, $\tilde{t} \approx 1.4$.

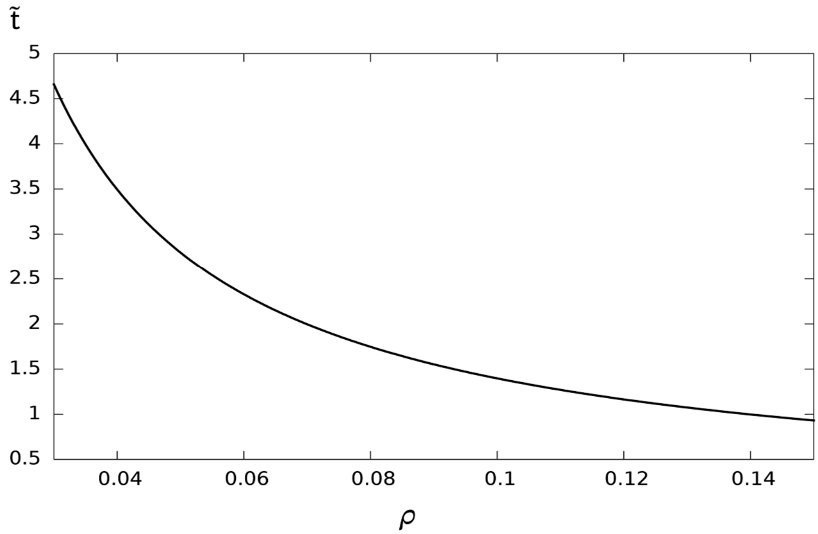

Figure 1: The relation between ρ and $\tilde{t}$

[5] In Mitchell et al. (2019) (pp. 357-358) it is stated that when $g > r$, there exists a stable steady state value of the debt-to-GDP ratio. Also see Wray (2016).

3.2. Relation between ρ and γ

Again we assume $d(0) = 0.45$. Figure 2 represents the relation between the value of ρ and the value of γ, which is the extra growth rate of the government expenditure by a fiscal policy, according to (3). The larger the value of ρ is, the larger the value of γ is. For example, when $\rho = 0.05$, $\gamma \approx 0.1$, when $\rho = 0.1$, $\gamma \approx 0.19$.

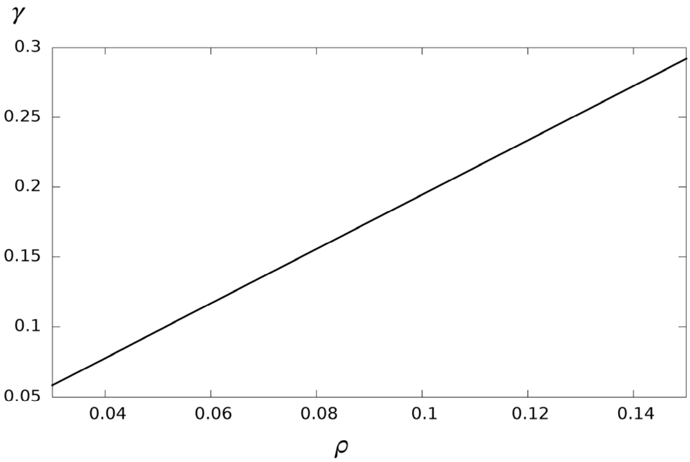

Figure 2: The relation between ρ and γ

3.3. Relation between ρ and $d(\tilde{t})$

We assume $d(0) = 0.45$. Figure 3 represents the relation between ρ and $d(\tilde{t})$, which is the debt-to-GDP ratio at the time when full employment is realized, according to (5). The larger the value of ρ is, the smaller the value of $d(\tilde{t})$ is, that is, the smaller the debt-to-GDP ratio at the time when full employment is realized.

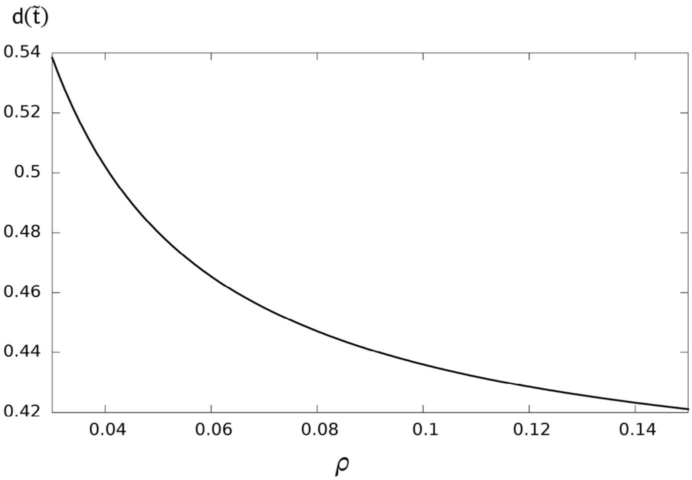

Figure 3: The relation between ρ and $d(\tilde{t})$

3.4. Relation between ρ and $d(\tilde{t})-d(0)$

We assume $d(0)=0.45$. Figure 4 represents the relation between ρ and $d(\tilde{t})-d(0)$, which is the difference between the debt-to-GDP ratio at $\tilde{t}$ and that at $t=0$, according to (6). The larger the value of ρ is, the smaller the value of $d(\tilde{t})-d(0)$ is. If ρ is larger than about 0.072, the debt-to-GDP ratio at $t=\tilde{t}$ is smaller than that at $t=0$, that is, the aggressive fiscal policy to realize full employment reduces the debt-to-GDP ratio.

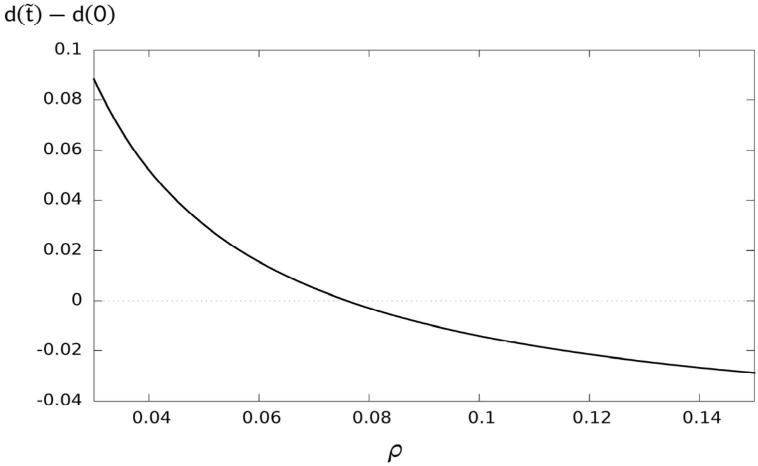

Figure 4: The relation between ρ and $d(\tilde{t})-d(0)$

3.5. Relation between t and $d(t)$

We assume $d(0)=0.45$ and $\rho=0.085$. Figure 5 represents the relation between the time (t) and the value of $d(t)$, which is the debt-to-GDP ratio at the time t, according to (4). First $d(t)$ increases, then it decreases.

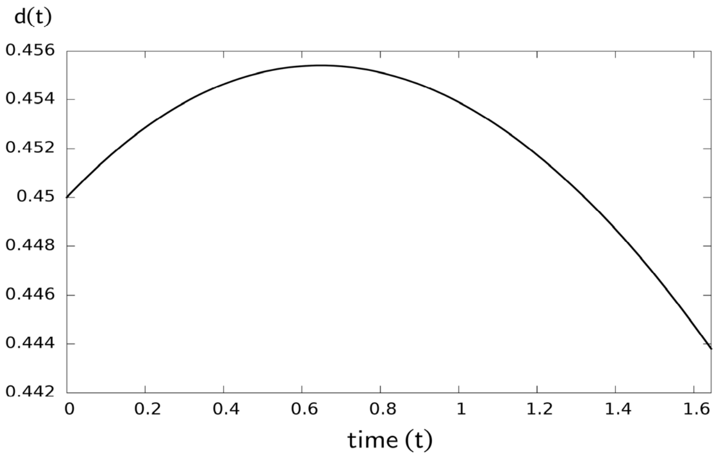

Figure 5: The relation between the time and $d(t)$

3.6. Relation between $\boldsymbol{t}$ and $\boldsymbol{d(t)-d(0)}$

Again we assume $d(0)=0.45$ and $\rho=0.085$. Figure 6 represents the relation between the time (t) and the value of $d(t)-d(0)$. First $d(t)-d(0)$ increases, then it decreases.

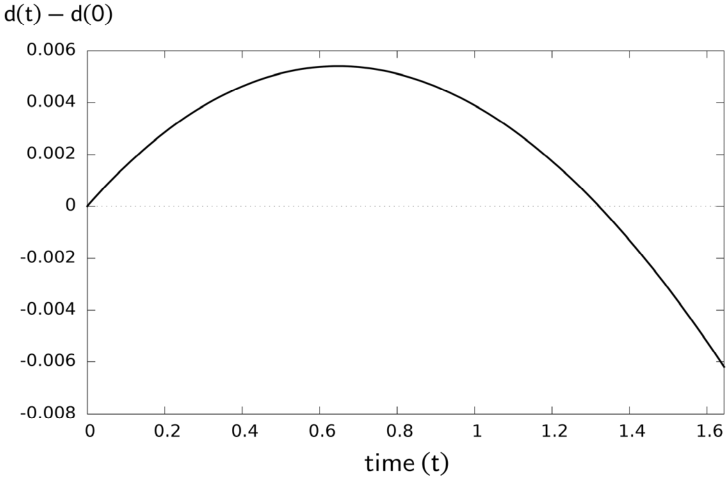

Figure 6: The relation between the time and $\boldsymbol{d(t)-d(0)}$

3.7. Relation between $\boldsymbol{d(0)}$ and $\boldsymbol{d(\tilde{t})}$

We assume $\rho=0.085$. Figure 7 represents the relation between the value of $d(0)$ and the value of $d(\tilde{t})$ according to (5). By (5) it is a straight line whose slope is smaller then one.

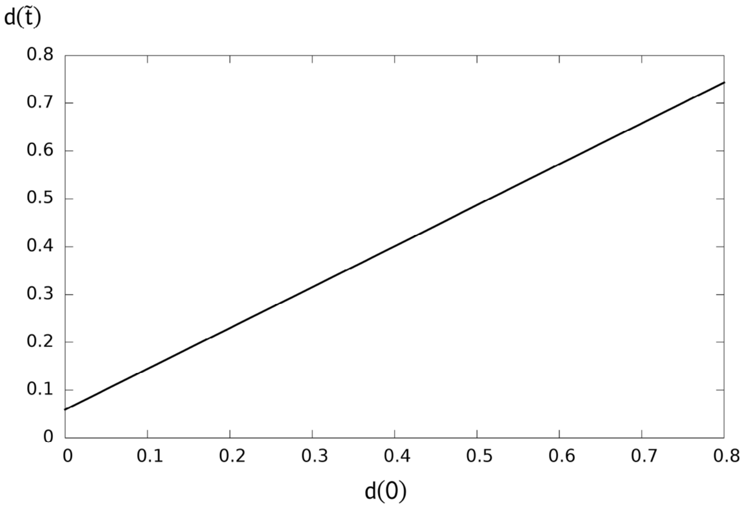

Figure 7: The relation between $\boldsymbol{d(0)}$ and $\boldsymbol{d(\tilde{t})}$

3.8. Relation between $\boldsymbol{d(0)}$ and $\boldsymbol{d(\tilde{t}) - d(0)}$

Again we assume $\rho = 0.085$. Figure 8 represents the relation between the value of $d(0)$ and the value of $d(\tilde{t}) - d(0)$ according to (6). By (6), since $e^{(r-g)\tilde{t}} < \zeta(= e^{\rho\tilde{t}})$, it is a straight line whose slope is negative.

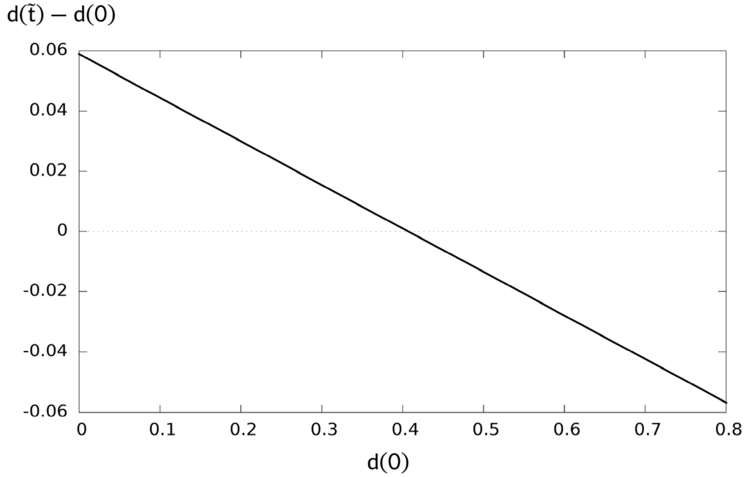

Figure 8: The relation between $\boldsymbol{d(0)}$ and $\boldsymbol{d(\tilde{t}) - d(0)}$

3.9. Relation between $\boldsymbol{\rho}$ and $\boldsymbol{d(\tilde{t}) - d(0)}$ with low and high interest rates

We assume $r = 0.035$. The values of other variables are the same as those in the previous cases. In Figure 9 we compare the relation between ρ and $d(\tilde{t}) - d(0)$ in the case of low interest rate and that in the case of high interest rate.

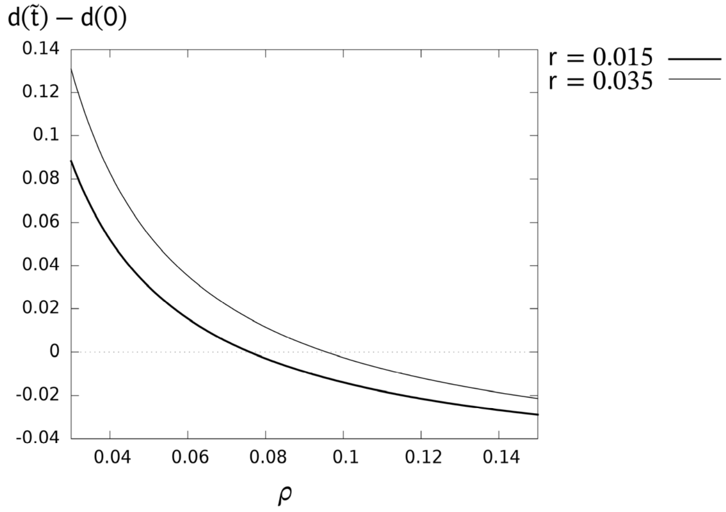

Figure 9: The relation between $\boldsymbol{\rho}$ and $\boldsymbol{d(\tilde{t}) - d(0)}$ with low and high interest rates

With higher interest rate the debt-to-GDP ratio at the time when full employment is realized is less likely smaller than that at time 0 than the case with low interest rate.

3.10. Relation between $\boldsymbol{\rho}$ and $\boldsymbol{d(\tilde{t}) - d(0)}$ with very small marginal propensity to consume

We assume $c = 0$. The values of other variables are the same as those in the previous cases. In Figure 10 we compare the relation between ρ and $d(\tilde{t}) - d(0)$ in this case and that when $c = 0.5$.

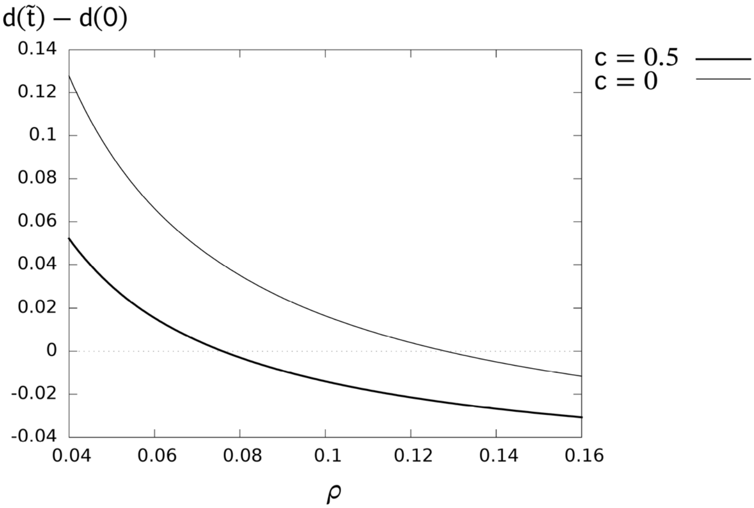

Figure 10: The relation between $\boldsymbol{\rho}$ and $\boldsymbol{d(\tilde{t}) - d(0)}$ in the case where $\boldsymbol{c = 0}$ and the case where $\boldsymbol{c = 0.5}$

Even if marginal propensity to consume is very small, an aggressive fiscal policy can reduce the debt-to-GDP ratio at the time when full employment is realized.

3.11. Relation between $\boldsymbol{\rho}$ and $\boldsymbol{d(\tilde{t}) - d(0)}$ when $\boldsymbol{d(0)}$ and $\boldsymbol{b(0)}$ have the steady state values

We assume $b(0) = (r - g)d(0)$. The values of other variables are the same as those in the previous cases. In Figure 11 we compare the relation between ρ and $d(\tilde{t}) - d(0)$ in this case and that $b(0) = -0.015$.

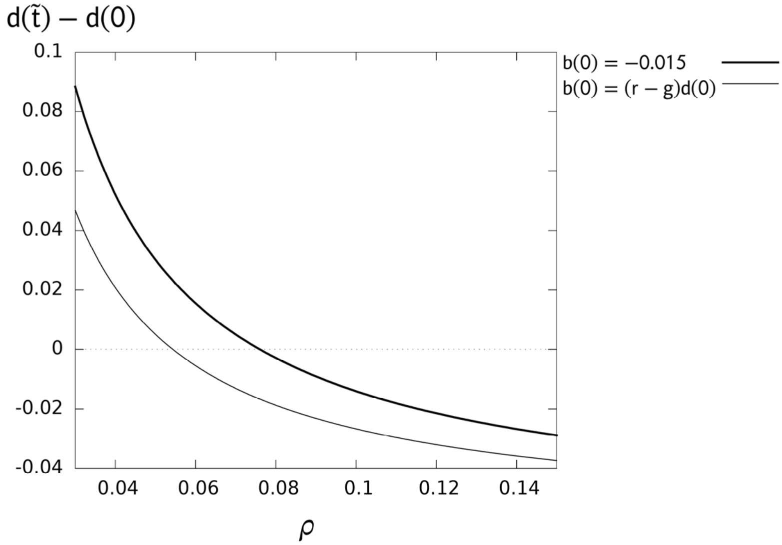

Figure 11: The relation between ρ and $d(\tilde{t}) - d(0)$ in the case where $b(0) = (r - g)d(0)$ and the case where $b(0) = -0.015$

If $d(0)$ and $b(0)$ have the steady state values, the debt-to-GDP ratio at the time when full employment is realized is more likely smaller than that at time 0 than the case where $b(0) = -0.015$. It is because $-0.015 < (r - g)d(0)$.

4. Concluding remarks

We have presented mathematical analyses and simulations of a fiscal policy which realizes full employment from an under-employment state without increasing the debt-to-GDP ratio than before the fiscal policy. We have shown the following results.

1. A fiscal policy to realize full employment from a state of under-employment can reduce the debt-to-GDP ratio.

2. The larger the extra growth rate (increasing rate) of real GDP by a fiscal policy is, the smaller the debt-to-GDP ratio at the time when full employment is realized is.

3. Even if the marginal propensity to consume is very small, by an appropriate fiscal policy we can realize full employment without increasing the debt-to-GDP ratio.

Also we considered a condition to realize full employment from a state of under-employment within one year without increasing debt-to-GDP ratio.

The main conclusion of this paper is that full employment can be realized by an aggressive fiscal policy with smaller debt-to-GDP ratio than before the fiscal policy.

Appendix: Derivation of multiplier by an overlapping generations model

We consider a two-period (young and old) overlapping generations model under monopolistic competition according to Otaki(2007, 2009). There is one factor of production, labor, and there is a continuum of goods indexed by $z \in [0,1]$. Each good is monopolistically produced by Firm z. Consumers are born at continuous density $[0,1] \times [0,1]$ in each period. They can supply only one unit of labor when they are young.

We use the following notations.

$c^i(z)$: consumption of good z at period i, $i = 1,2$.
$p^i(z)$: the price of good z at period i, $i = 1,2$.

$$X^i = \left\{\int_0^1 c^i(z)^{1-\frac{1}{\eta}} dz\right\}^{\frac{1}{1-\frac{1}{\eta}}}, \qquad i = 1,2, \qquad \eta > 1.$$

ξ: disutility of labor, $\xi > 0$.
$0 < \alpha < 1$.
W: nominal wage rate.
Π: profits of firms which are equally distributed to each consumer.
L: employment of each firm and the total employment.
L_f: population of labor or employment at the full employment state.
y: labor productivity, $y \geq 1$.
δ is the definition function. If a consumer is employed, $\delta = 1$; if he is not employed, $\delta = 0$. The labor productivity is y, that is, y unit of the goods is produced by one unit of labor. The utility of consumers of one generation over two periods is

$$U(X^1, X^2, \delta, \xi) = (X^1)^{\alpha}(X^2)^{1-\alpha} - \delta\xi, \ 0 < \alpha < 1 \ .$$

With the budget constraint

$$\int_0^1 p^1(z)c^1(z)dz + \int_0^1 p^2(z)c^2(z)dz = \delta W + \Pi.$$

$p^2(z)$ is the expectation of the price of good z at period 2. The Lagrange function is

$$\mathcal{L} = (X^1)^{\alpha}(X^2)^{1-\alpha} - \delta\xi - \lambda\left(\int_0^1 p^1(z)c^1(z)dz + \int_0^1 p^2(z)c^2(z)dz - \delta W - \Pi\right).$$

λ is the Lagrange multiplier. The first order conditions are

$$\alpha(X^1)^{\alpha-1}(X^2)^{1-\alpha}\left\{\int_0^1 c^1(z)^{1-\frac{1}{\eta}} dz\right\}^{\frac{\frac{1}{\eta}}{1-\frac{1}{\eta}}} c^1(z)^{-\frac{1}{\eta}} = \lambda p^1(z),$$

and

$$(1-\alpha)(X^1)^{\alpha}(X^2)^{-\alpha}\left\{\int_0^1 c^2(z)^{1-\frac{1}{\eta}} dz\right\}^{\frac{\frac{1}{\eta}}{1-\frac{1}{\eta}}} c^2(z)^{-\frac{1}{\eta}} = \lambda p^2(z).$$

They are rewritten as

$$\alpha(X^1)^{\alpha}(X^2)^{1-\alpha}\left\{\int_0^1 c^1(z)^{1-\frac{1}{\eta}}dz\right\}^{-1} c^1(z)^{1-\frac{1}{\eta}} = \lambda p^1(z)c^1(z), \quad (7)$$

and

$$(1-\alpha)(X^1)^{\alpha}(X^2)^{1-\alpha}\left\{\int_0^1 c^2(z)^{1-\frac{1}{\eta}}dz\right\}^{-1} c^2(z)^{1-\frac{1}{\eta}} = \lambda p^2(z)c^2(z). \quad (8)$$

From (7) and (8) we obtain

$$\alpha(X^1)^{\alpha}(X^2)^{1-\alpha}\left\{\int_0^1 c^1(z)^{1-\frac{1}{\eta}}dz\right\}^{-1} \int_0^1 c^1(z)^{1-\frac{1}{\eta}}dz$$
$$= \alpha(X^1)^{\alpha}(X^2)^{1-\alpha} = \lambda \int_0^1 p^1(z)c^1(z)dz,$$

and

$$(1-\alpha)(X^1)^{\alpha}(X^2)^{1-\alpha}\left\{\int_0^1 c^2(z)^{1-\frac{1}{\eta}}dz\right\}^{-1} \int_0^1 c^2(z)^{1-\frac{1}{\eta}}dz$$
$$= (1-\alpha)(X^1)^{\alpha}(X^2)^{1-\alpha} = \lambda \int_0^1 p^2(z)c^2(z)dz.$$

Thus, we get

$$\frac{\int_0^1 p^1(z)c^1(z)dz}{\int_0^1 p^2(z)c^2(z)dz} = \frac{\alpha}{1-\alpha},$$

$$\int_0^1 p^1(z)c^1(z)dz = \alpha(\delta W + \Pi),$$

and

$$\int_0^1 p^2(z)c^2(z)dz = (1-\alpha)(\delta W + \Pi).$$

Therefore, the aggregate demand of the younger generation is

$$\alpha(\delta W + \Pi).$$

The total aggregate demand is

$$\alpha(WL + \Pi) + G + M.$$

G is the government expenditure and M is consumption by the old generation. Since in the model of this appendix the goods are produced by only labor, the investments by firms are zero. The aggregate supply is

$$P^1Ly = WL + \Pi.$$

The profit of a firm is written as

$$\Pi = P^1Ly - WL.$$

Since the aggregate demand and supply are equal,

$$P^1Ly = \alpha P^1Ly + G + M.$$

In real terms

$$Ly = \frac{1}{1-\alpha}\left(\frac{G}{P^1} + \frac{M}{P^1}\right).$$

Therefore, we get the multiplier $\frac{1}{1-\alpha} > 0$.

Acknowledgement

We are very grateful to the referee for useful comments which substantially improved this paper.

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